

Quantum-Mechanical Concepts in the Waveguides Theory

N. Marinescu

Department of Physics, University of Bucharest, Romania

M. Apostol

Department of Theoretical Physics, Institute for Atomic Physics, Măgurele – Bucharest MG-6,
P.O. Box MG-35, Romania

Z. Naturforsch. **47a**, 935–940 (1992); received July 19, 1991

A Klein-Gordon-type equation is derived for the wave propagation in an ideal, uniform waveguide, and its quantum-mechanical interpretation is given. The “cross-section” concept is introduced for a waveguide and the power transmission factor is obtained by using standard methods of quantum mechanics.

The spinorial formalism is also employed for deriving the equivalent Dirac-type equation, and the perturbation theory is applied for computing the frequency shifts. The general applicability of the quantum-mechanical concepts to the waveguides theory is discussed.

1. Introduction

Beside the standard methods of the e.m. field theory [1, 2], new methods, borrowed from quantum mechanics and field theory, have recently gained ground [3, 4] in studying the propagation of the e.m. waves in waveguides and resonant cavities. It has recently been shown [5] that the propagation of the e.m. wave through an ideal, uniform waveguide can be described by a wave-equation in $1+1$ dimensions with a potential barrier standing for the waveguide. This potential barrier arises by freezing out the degrees of freedom corresponding to the transversal motion (i.e., motion along directions perpendicular to the waveguide axis), as corresponding to the transversal stationary waves. It may also be interpreted as a mass-term (depending on the eigenfrequencies of the waveguide modes) of a Klein-Gordon-type equation.

This method is further developed in the present paper, with the aim of enlarging the applicability of some quantum-mechanical concepts to the wave propagation in the waveguides. Specifically, the number of particles and the current operators are introduced for the field operator (of positive frequencies), based on the continuity equation. This allows one to compute, by using standard methods of quantum mechanics, that part of the wave function which is “scattered” by the waveguide. In this connection one may introduce the notion of the “cross-section” of a

waveguide and one can derive the power transmission factor of a waveguide. In order to treat the interaction of an e.m. wave propagating through a waveguide, as well as to deepen the physical picture, the equivalent Dirac-type equation is derived, and the corresponding spinorial field is established. Perturbing elements, as small-size objects, dielectric or plasma media, etc., are treated within the first-order perturbation theory, in order to get the corresponding frequency shifts brought about by these elements, and to illustrate the way in which this formalism works.

2. Klein-Gordon-Type Equation for Waveguides

It is well-known that there are two distinct types of e.m. waves propagating in a waveguide: transverse electric (TE) and transverse magnetic (TM) waves. For the former the component of the electric field parallel to the waveguide axis is zero ($E_x=0$), while for the latter the same component of the magnetic field ($H_x=0$) vanishes. Assuming an ideal, uniform waveguide the e.m. field of TM-type can be expressed in terms of the E_x component in the following way [6]:

$$\begin{aligned} E_y &= i \frac{c^2 p}{\omega_0^2} \frac{\partial E_x}{\partial y}, & H_y &= -i \frac{c \omega}{\omega_0^2} \frac{\partial E_x}{\partial z}, \\ E_z &= i \frac{c^2 p}{\omega_0^2} \frac{\partial E_x}{\partial z}, & H_z &= i \frac{c \omega}{\omega_0^2} \frac{\partial E_x}{\partial y}, \end{aligned} \quad (1)$$

Reprint requests to N. Marinescu, Department of Physics, University of Bucharest, Bucharest, Romania.

0932-0784 / 92 / 0900-0935 \$ 01.30/0. – Please order a reprint rather than making your own copy.



Dieses Werk wurde im Jahr 2013 vom Verlag Zeitschrift für Naturforschung in Zusammenarbeit mit der Max-Planck-Gesellschaft zur Förderung der Wissenschaften e.V. digitalisiert und unter folgender Lizenz veröffentlicht: Creative Commons Namensnennung-Keine Bearbeitung 3.0 Deutschland Lizenz.

Zum 01.01.2015 ist eine Anpassung der Lizenzbedingungen (Entfall der Creative Commons Lizenzbedingung „Keine Bearbeitung“) beabsichtigt, um eine Nachnutzung auch im Rahmen zukünftiger wissenschaftlicher Nutzungsformen zu ermöglichen.

This work has been digitalized and published in 2013 by Verlag Zeitschrift für Naturforschung in cooperation with the Max Planck Society for the Advancement of Science under a Creative Commons Attribution-NoDerivs 3.0 Germany License.

On 01.01.2015 it is planned to change the License Conditions (the removal of the Creative Commons License condition “no derivative works”). This is to allow reuse in the area of future scientific usage.

where

$$\left(A_{y,z} + \frac{\omega_0^2}{c^2} \right) E_x = 0, \quad E_x|_r = 0 \quad (2)$$

and $\omega = \sqrt{c^2 p^2 + \omega_0^2}$. In these equations c is the light velocity, p is the wavevector along the x -director (waveguide axis), ω_0 is the eigenfrequency of a waveguide mode determined by (2) and Γ is the boundary of the transversal section of the waveguide. The coordinates along the directions perpendicular to the waveguide axis are denoted by y and z .

In addition a space-time dependence of the form $\exp[i(p x - \omega t)]$ has been assumed for the e.m. field. Similar equations hold for the TE-type, the boundary condition being in this case $\text{grad}_{y,z} H_x|_r = 0$.

The TM-field given by (1) and (2) can be quantized (for a fixed ω_0 -mode) by

$$E_x = \sum_p \sqrt{\frac{2\pi}{\omega}} \omega_0 f(y, z) [a_p e^{i(p x - \omega t)} + a_p^+ e^{-i(p x - \omega t)}], \quad (3)$$

where a_p and a_p^+ are the annihilation, and respectively, creation operators of the p -photon state, satisfying the commutation rules

$$[a_p, a_{p'}^+] = \delta_{p p'}, \quad [a_p, a_p] = 0 \quad (\hbar = 1),$$

and the function $f(y, z)$ is given by (2) with the normalization condition

$$\int f^2(y, z) dy dz = 1,$$

the integration being extended to the whole transverse section of the waveguide. Making use of (3) one can easily obtain the energy content of the unit volume

$$W = \frac{1}{8\pi} \int (E^2 + H^2) d\mathbf{r} = \sum_p \omega a_p^+ a_p \quad (4)$$

and the energy flow

$$S = \frac{c}{4\pi} \int (\mathbf{E} \times \mathbf{H})_x d\mathbf{r} = \sum_p c n \omega \frac{p}{|p|} a_p^+ a_p, \quad (5)$$

which tell us that the e.m. field in the waveguide may be viewed as consisting of photons of energy ω which carry this energy with the group velocity $c n$,

$$n = \sqrt{1 - \omega_0^2/\omega^2} \quad (6)$$

being the refractive index of the waveguide.

The same picture can be attained by using the Klein-Gordon-type equation

$$\frac{\partial^2 \Psi}{\partial t^2} = \left(c^2 \frac{\partial^2}{\partial x^2} - \omega_0^2 \right) \Psi, \quad (7)$$

where the field Ψ , given by

$$\Psi(x, t) = \sum_p \frac{1}{\sqrt{2\omega}} a_p e^{i(p x - \omega t)}, \quad (8)$$

is constructed with the same photon operators a_p as the e.m. field (3) but contains only the positive-frequency part of this field. Indeed, the hamiltonian corresponding to (7) is easily obtained as

$$W = \int \left(\frac{\partial \Psi^+}{\partial t} \cdot \frac{\partial \Psi}{\partial t} + c^2 \frac{\partial \Psi^+}{\partial x} \frac{\partial \Psi}{\partial x} + \omega_0^2 \Psi^+ \Psi \right) dx \\ = \sum_p \omega a_p^+ a_p, \quad (9)$$

and from the continuity equation

$$\frac{\partial}{\partial t} \left(\Psi^+ \frac{\partial \Psi}{\partial t} - \frac{\partial \Psi^+}{\partial t} \Psi \right) \\ - c^2 \frac{\partial}{\partial x} \left(\Psi^+ \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^+}{\partial x} \Psi \right) = 0 \quad (10)$$

one gets straightforwardly the number of photons per unit volume

$$N = i \int \left(\Psi^+ \frac{\partial \Psi}{\partial t} - \frac{\partial \Psi^+}{\partial t} \Psi \right) dx = \sum_p a_p^+ a_p \quad (11)$$

and the current

$$J = -i c^2 \int \left(\Psi^+ \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^+}{\partial x} \Psi \right) dx \\ = \sum_p c n \frac{p}{|p|} a_p^+ a_p, \quad (12)$$

which tells that the photons in the waveguide may be viewed as particles propagating with the group velocity $c n$. The well-known critical condition for propagating waves in the waveguides, $\omega > \omega_0$, is readily obtained from (6) and (12).

The ω_0 -term in the Klein-Gordon equation (7) may be viewed either as a mass-term or as a potential barrier (for the wave-equation) with non-vanishing values over the whole space occupied by the waveguide. Eq. (7) describes the propagation of the e.m. wave through the waveguide for each wave-mode ω_0 , and as such, the field Ψ depends on ω_0 . We stress that the field Ψ given by (8), although expressed with the photon operators a_p , should not be interpreted as the photon wavefunction since, for example, a probability density of photon localization is meaningless [7]. It

rather corresponds to a number of N photons per unit volume with energy $\omega = \sqrt{c^2 p^2 + \omega_0^2}$ propagating in the waveguide with the group velocity cn . The corresponding e.m. field can be obtained in terms of Ψ by using (3) and (1), via the photon operators a_p .

3. "Cross-Section" of a Waveguide and the Power Transmission Factor

For a wave of frequency ω propagating along a waveguide, (7) reads

$$\left(c^2 \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right) \varphi(x) e^{-i\omega t} = \omega_0^2 \varphi(x) e^{-i\omega t}. \quad (13)$$

In the absence of the waveguide ($\omega_0 = 0$) the source-term in (13) disappears and the solution is the plane-wave $\exp[i(p x - \omega t)]$. We may ask what is this in-going wave changed into by the waveguide, i.e. what is the wave "scattered" by the waveguide. Assuming that the waveguide may be viewed as a perturbation, and limiting ourselves to the first-order of the perturbation theory (Born approximation), (13) becomes

$$\left(c^2 \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right) \varphi(x) e^{-i\omega t} \cong \omega_0^2 \varphi e^{i(p x - \omega t)}, \quad (14)$$

where ω_0 is viewed as a "scattering" potential extended over the length L of the waveguide.

The retarded Green function of (14) is given by

$$G(x, t) = \frac{1}{(2\pi)^2} \int dp d\omega (\omega^2 - c^2 p^2)^{-1} e^{i(p x - \omega t)}, \quad (15)$$

where the $\omega = \pm c p$ poles are pushed down into the lower half-plane of the complex variable ω . The integration path in (15) is shown in Figure 1. Following this rule one may easily perform the integration in (15) to get

$$G(x, t) = \frac{1}{4c} [\theta(x + ct) - \theta(x - ct)], \quad (16)$$

where $t > 0$ ($G(x, t < 0) = 0$) and θ is the step-function. The solution of (14) is easily obtained now as

$$\varphi(x) = -i \frac{\omega_0^2}{4\omega c} f\left(k - \frac{\omega}{c} \operatorname{sgn}(x)\right) e^{i\omega/c |x|}, \quad (17)$$

where

$$f(p) = (e^{ipL} - 1)/ip$$

is the form-factor of the waveguide. For $L/\lambda \ll 1$, where $\lambda = 2\pi/p$, one may approximate $f(p)$ by L .

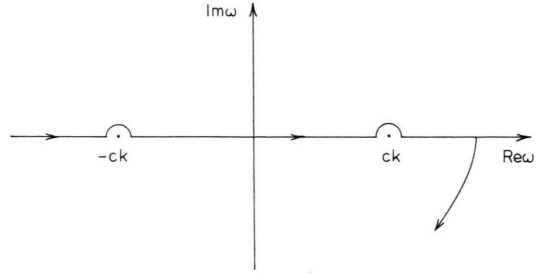


Fig. 1. Integration path for the retarded Green function given by (15).

Note that while the wave-function in (13) contains only the positive frequency part the "scattered" wave function includes both positive- and negative-frequency contributions $\omega = \pm c p$ (through the poles of the Green function), as corresponding to the "transmitted" ($x > 0$) and the "reflected" wave in (17).

The amplitude of the "scattered" wave can also be obtained in a more straightforward way. One can estimate the extend to which the wave $\exp\left(i \frac{\omega}{c} n x\right)$, propagating in the waveguide, is present in the incoming wave $\exp\left(i \frac{\omega}{c} x\right)$. The answer is given by the amplitude:

$$\begin{aligned} A &= \frac{1}{L} \int_0^L dx \exp\left[i \frac{\omega}{c} (n-1) x\right] \\ &= \left[iL \frac{\omega}{c} (n-1)\right]^{-1} \left\{ \exp\left[i \frac{\omega}{c} (n-1) L\right] - 1 \right\}. \end{aligned} \quad (19)$$

For $L/\lambda \ll 1$ one may expand A given by (19) as

$$A = 1 + i \frac{\omega}{2c} (n-1) L + \dots \quad (20)$$

and, making use of n given by (6), one gets

$$A \cong 1 - i \frac{\omega_0^2}{4\omega c} L + \dots \quad (21)$$

One can see that the "scattered" amplitude of the wave-function $\varphi(x)$ given by (17) is exactly $A - 1$, providing the form-factor $f(p)$ in (17) is approximated by L . The current of photons propagating through the waveguide is easily obtained from (12) and (17),

$$J = c(\omega_0^2 L / 4\omega c)^2, \quad (22)$$

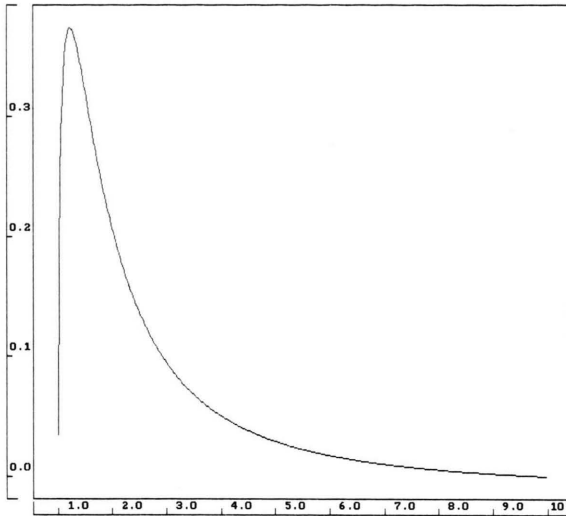


Fig. 2. The transmission factor t given by (25) vs. eigenfrequency ω_0 . Note the maximum “transparency” of the waveguide mode $\omega_0 = (2/\sqrt{5})\omega$.

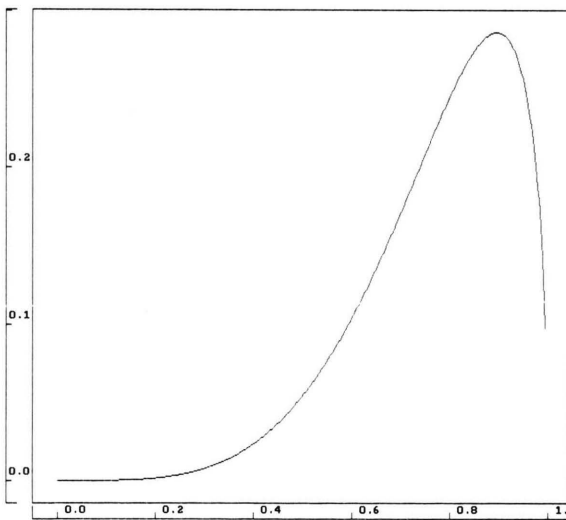


Fig. 3. The transmission factor t given by (25) vs. frequency ω of the incoming wave, for a fixed eigenfrequency ω_0 . Note the evanescence condition $\omega < \omega_0$.

which, compared to the in-coming current $J_0 = c$, allows one to define the “cross-section” of the waveguide

$$\sigma = J/J_0 = (\omega_0^2 L/4\omega c)^2 \quad (23)$$

as representing the fraction of the in-coming flux of particles (number of photons per unit time and unit

area) propagating through the waveguide and carrying the energy ωn per particle. One can easily derive from (23) the ratio of the transmitted power $P = \omega n J$ to the in-coming power $P_0 = \omega J_0$, i.e. the power transmission factor

$$t = P/P_0 = \sigma n. \quad (24)$$

Making use of (6) and (23) one readily obtains

$$t = (\omega_0^2 L/4\omega c)^2 \sqrt{1 - \omega_0^2/\omega^2}. \quad (25)$$

One has to emphasize that (25) is valid for $L/\lambda \ll 1$ and for $\omega_0/\omega \ll 1$, the latter condition being required by the Born approximation. Nevertheless, we shall use the expression of t given by (25) for any ω_0 , for the purpose of illustrating its physical meaning.

One can see, from (23) and (25), that not the whole beam of photons is transmitted by the waveguide, but only a fraction of it; the remainder is partly reflected and partly absorbed by the stationary waves which built up in the waveguide. The transmission factor t given by (25) is plotted in Fig. 2 as function of the eigenfrequency ω_0 for a fixed frequency ω of the incoming wave. One can see that there is no transmission through the waveguide for $\omega < \omega_0$, and that the waveguide has the maximum “transparency” for the eigenfrequency $\omega_0 = (2/\sqrt{5})\omega$. One may also ask what is the frequency ω most penetrating through a given ω_0 -mode. The answer is $\omega = \sqrt{3}/2 \cdot \omega_0$, as representing the frequency for which the function $t(\omega)$ plotted in Fig. 3 reaches its maximum value. The total fraction of the in-coming power transmitted through all the eigenfrequencies ω_0 of the waveguide may also be easily computed by counting all the eigenmodes corresponding to the eigenfrequency range $(\omega_0, \omega_0 + d\omega_0)$; their number is $(2S/\pi c^2)\omega_0 d\omega_0$, where S is the transversal area of the waveguide, so that the total power fraction transmitted through the waveguide is obtained by integrating (25) with this weight. One obtains

$$\begin{aligned} t_{\text{tot}} &= \int_0^\omega d\omega_0 (2S/\pi c^2) \omega_0 t(\omega_0) \\ &= \frac{4VL\omega^4}{105c^4} = \frac{64\pi^4 VL}{105\lambda^4}, \end{aligned} \quad (26)$$

where V denotes the volume of the waveguide and λ stands for the wavelength of the in-coming wave. We recall that (26) holds only for $LV/\lambda^4 \ll 1$.

4. Dirac-Type Equation for the Waveguides

Of particular importance for the physics of the waveguides is the propagation of the e.m. waves in the presence of perturbing elements as dielectric media, plasmas, laser beams, etc. In this case it is more convenient to get a linear equation in ω (first-order derivative equation) instead of (7). This can be achieved by splitting (7) up, as it is usually done in relativistic quantum mechanics for obtaining the Dirac equation [8]. Following the same procedure, one obtains

$$i \frac{\partial \Psi}{\partial t} = H \Psi, \quad (27)$$

where

$$H = c \alpha p + \omega_0 \beta \quad (28)$$

is the Dirac-type hamiltonian for a two-component spinor Ψ , and the matrices α and β may be chosen as

$$\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (29)$$

The solutions of this equation are readily obtained as $u_p e^{ipx}$ and $v_p e^{ipx}$, where the positive-frequency spinor u_p is given by

$$u_p = [(\omega + \omega_0)/2\omega]^{1/2} \cdot \begin{pmatrix} 1 \\ \frac{cp}{\omega + \omega_0} \end{pmatrix}, \quad (30)$$

and the negative-frequency spinor v_p is given by

$$v_p = [(\omega + \omega_0) 2\omega]^{1/2} \cdot \begin{pmatrix} -\frac{cp}{\omega + \omega_0} \\ 1 \end{pmatrix}, \quad (31)$$

with $\omega = \sqrt{c^2 p^2 + \omega_0^2}$. However, the spinor describing the photon propagation in the waveguide is a certain superposition of u_p and v_p which may be written as

$$w_p = \begin{pmatrix} \sqrt{\frac{\omega + \omega_0}{2\omega}} \operatorname{sgn}(p) \\ \sqrt{\frac{\omega - \omega_0}{2\omega}} \end{pmatrix}. \quad (32)$$

Indeed, the field operator

$$\Psi(x) = \sum_p w_p a_p e^{ipx} \quad (33)$$

yields the correct energy content per unit volume

$$W = \int \Psi^\dagger(x) H \Psi(x) dx = \sum_p \omega a_p^\dagger a_p, \quad (34)$$

and the number of photons

$$N = \int \Psi^\dagger(x) \Psi(x) dx = \sum_p a_p^\dagger a_p \quad (35)$$

carrying this energy with the group velocity cn , as one can see from the continuity equation

$$\frac{\partial}{\partial t} (\Psi^\dagger \Psi) + c \frac{\partial}{\partial x} (\Psi^\dagger \alpha \Psi) = 0 \quad (36)$$

corresponding to (27) and from the current

$$J = c \int \Psi^\dagger \alpha \Psi dx = \sum_p cn \frac{p}{|p|} a_p^\dagger a_p. \quad (37)$$

Of course, one can not speak of the photon wave-function Ψ but, rather, of the Dirac-type field Ψ given by (33) as describing the photon propagation in the waveguide. We note also that the photon operators a_p appearing in (33) are the same as those in terms of which the e.m. field (3) is expressed.

A perturbation U to the hamiltonian (28) will, generally, split off the two degenerate levels labelled by $\pm p$, so that, to the first-order of the perturbation theory the frequency shift $\delta\omega$ is given by the secular equation

$$|\int \varphi_p^\dagger U \varphi_{p'} dx - \delta\omega \delta_{pp'}| = 0, \quad (38)$$

where $\varphi_p = w_p e^{ipx}$, and p and p' run over the $\pm p$ values. If the perturbing potential U extends over a length d , then a factor d/L occurs in the diagonal elements of the perturbation matrix and, in the limit $d/\lambda \ll 1$, the same factor appears in the off-diagonal elements too. A similar geometric factor χ should also be included in the matrix elements of U , as due to the corresponding waveguide mode ω_0 . For the fundamental mode and a perturbing element extending over the transverse area $s \ll S$, χ may be approximated by s/S , where S is the transverse area of the waveguide. Under these circumstances a factor $v/V \ll 1$ should therefore be included in the matrix elements of U , where v is the volume of the perturbing element and V is the waveguide volume. In order to carry out further the calculations one has to specify the perturbation U . For a dielectric of constant ε (or a plasma) and eigenfrequency ω_c , one obtains from (6)

$$\omega_c^2 = \omega^2 n_0 (1 - n^2/n_0^2) = \omega^2 n_0^2 (1 - \varepsilon/\varepsilon_0),$$

where n_0 and ε_0 are the refractive index and, respectively, the dielectric constant of the unperturbed medium of the waveguide. Therefore, the perturbation reads in this case

$$U = \frac{1}{2} \omega n_0^2 (1 - \varepsilon/\varepsilon_0), \quad (39)$$

which yields, according to (38), the frequency shift

$$\frac{\delta\omega}{\omega} = \frac{v}{2V} n_0^2 (1 - \varepsilon/\varepsilon_0) (1 \pm \omega_0/\omega), \quad (40)$$

the ω_0/ω -splitting factor arising from the $W_p^+ w_{-p}$ contribution of the w_p -spinors (32) to the off-diagonal matrix elements. This reflects again the fact that the photon corresponds to a superposition of positive- and negative-frequency parts. In the case of the $p=0$ mode this contribution disappears.

5. Conclusions

In conclusion, one may say that the propagation of the e.m. waves through a waveguide can be described in a very convenient manner by using well-known quantum-mechanical concepts. It has been shown that a field operator can be introduced which has the correct interpretation as regards the photon energy, number and current, according to the continuity equation. Moreover, this field operator is constructed with the photon operators corresponding to the momentum eigenstates and, therefore, it can be used to express the corresponding e.m. field. The equation satisfied by this field is either the Klein-Gordon-equation or the Dirac equation (with a two-component spinor), which are amenable to the standard methods of relativistic quantum mechanics. The mass-term in these equations may also signify the potential barrier representing the waveguide, originating in the eigenfrequencies of the waveguide modes. Within this for-

malism one readily recovers the usual terms of the standard description of the waveguides physics and one is also able to introduce new concepts as the waveguide "cross-section", "scattered" wavefunction power transmission factor, etc. The simplest case of an ideal uniform waveguide has been discussed in order to illustrate the formalism. The case of a power loss waveguide can be tackled similarly by adequately extending the present approach.

Of particular importance in this respect is the interacting e.m. waves propagating in waveguides or resonant cavities. According to the present picture these waves may be viewed as corresponding to relativistic (uncharged) massive fields governed by a Dirac-type equation in 1+1 dimensions (or in 2+1 dimensions, as for e.m. waves propagating between two reflecting plates), the corresponding spinors having two components and being a certain superposition of positive- and negative-frequency parts. The perturbation theory of the interacting fields can be applied to this situation in order to solve the own problems of waveguides physics or to formulate new ones. The interaction of the abovementioned fields with the polarization waves in a dielectric medium, or with the plasma oscillations in a medium introduced in the waveguide, or the interaction with the e.m. waves of a coherent laser beam, can be interesting situations amenable to such a treatment. Interesting new objects can appear in this new context of the "quantum" waveguides.

For example, a resonant cavity may be viewed as a gas of stationary (non-interacting) boson-states, with various masses, corresponding to the eigenfrequencies of the cavity. One may note also that the present formalism is applicable to optical fibres too, viewed as waveguides. Various applications of this method are in progress and will be published in forthcoming papers.

- [1] C. G. Montgomery, R. H. Dicke, and E. M. Purcell, *Principles of Microwaves Circuits*, MIT Radiation Laboratory Series, Vol. 8, McGraw-Hill Book Company, Inc., New York 1948, pp. 33.
- [2] D. S. Jones, *The Theory of Electromagnetism*, 64, Pergamon Press, Oxford 1964.
- [3] *Resonances-Models and Phenomena-Proceedings Bielefeld 1984*. Edited by S. Albeverio, L. S. Ferreira, and L. Streit.
- [4] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics*, Springer, Berlin 1988.
- [5] N. Marinescu and R. Nistor, *Quantum Features of Microwave Propagation in a Rectangular Waveguide*, *Z. Naturforsch.* **45a**, 953 (1990).
- [6] L. Landau and E. Lifschitz, *Electrodynamique des Milieux Continus*, Edition MIR, Moscova 1969.
- [7] H. Feshbach and F. Villars, *Rev. Mod. Phys.* **30**, 24 (1958).
- [8] A. Messiah, *Mecanique Quantique*, Dunod, Paris 1985.